MTH 520/622 Final Solutions

1. Let $\mathscr{C}_A$ be the reflection about a circle $A \in \hat{\mathbb{C}}$.

   (a) Derive an explicit formula for $\mathscr{C}_A$.

   (b) Consider $f_{A_1,A_2} = \mathscr{C}_{A_1} \circ \mathscr{C}_{A_2}$, where A_1, A_2 are distinct circles in $\hat{\mathbb{C}}$. Show that

      (i) If A_1 and A_2 are ultra-parallel, then f_{A_1,A_2} is loxodromic.

      (ii) If A_1 and A_2 meet at ∞, then show that f_{A_1,A_2} is parabolic.

      (iii) If A_1 and A_2 meet are intersecting, then show that f_{A_1,A_2} is elliptic.

   (c) Show every $m \in \text{Möb}^+(\mathbb{H})$ is a composition of at most three reflections. [Hint: Any isometry is uniquely determined by where it takes 3 non-collinear points. Consider the hyperbolic triangle determined by these points.]

   Solution. (a) The complex inversion $\mathscr{C}_A$ (i.e. reflection) about the unit circle $A \subset \hat{\mathbb{C}}$ is given by $\mathscr{C}_A(z) = 1/\bar{z}$. Consequently, the reflection about the Euclidean circle $A' = \{z \in \mathbb{C} : |z - z_0| = r\}$ is given by

$$\mathscr{C}_{A'}(z) = \frac{r^2}{\bar{z} - \bar{z}_0} + z_0.$$

   Suppose that B is Euclidean line making an angle θ with $\mathbb{R}$ at $z = a$, and $A'' = B \cup \{\infty\}$. Then a direct calculation would reveal that

$$\mathscr{C}_{A''} = e^{-i\theta}(\bar{z} - a) + a.$$

   (b) (i) Let A_1 and A_2 be ultraparallel lines in $\mathbb{H}$ (i.e they do not meet wither in $\mathbb{H}$ or $\partial\,\mathbb{H}$.) Then we know from class that there is a unique geodesic A that is perpendicular to both A_1 and A_2, which realizes the distances between them. Clearly, f_{A_1,A_2} has to preserve A (why?) and and its end points $\partial\,\mathbb{H}$. Hence, f_{A_1,A_2} is loxodromic, and A is the geodesic axis of f_{A_1,A_2}.

   (ii) If A_1 and A_2 meet at ∞, then f_{A_1,A_2} has a unique fixed point (namely ∞) in $\partial\,\mathbb{H}$. Hence, f_{A_1,A_2} is parabolic.

(iii) Suppose that A_1 and A_2 intersect at a point $P \in \mathbb{H}$. Then P is a unique fixed point (in $\mathbb{H}$) of f_{A_1,A_2}, which makes it elliptic.

(c) Let $m \in \mathrm{Isom}^+(\mathbb{H})$. Choose three non-collinear points $A, B, C \in \mathbb{H}$ and $m(A), m(B), m(C)$. If $A = m(A)$ and $B = m(B)$, then the reflection $\mathscr{C}$ in the line through A and B maps C to $m(C)$. Hence, $\mathscr{C} = m$.

Suppose that $A = m(A)$. Then consider the reflection $\mathscr{C}'$ in the line through A that is equidistant from both B and $m(B)$. Clearly, $B = m(B)$, and if $C = m(C)$ we are done. However, if $C \neq m(C)$, we compose $\mathscr{C}'$ with a reflection in the line through $m(A)$ and $m(B)$ to conclude that m is a composition of at most two reflections. By generalizing this argument, we obtain the desired result.

2. Show every $m \in \text{Möb}^+(\mathbb{H})$ is a composition of at most three reflections.

   (a) Show that the three angle bisectors of T meet at a single point.

   (b) If each side of T has the same length, then show that interior angles of T are equal. Moreover, if α is the interior of T at a vertex and a is the length of each side, then prove that

$$2\cosh\left(\frac{a}{2}\right)\sin\left(\frac{\alpha}{2}\right) = 1.$$

Solution. (a) Let A, B and C be the vertices of the hyperbolic triangle T, and let α, β, and γ be the internal angles at these vertices. Any angle bisector of T is the unique (why?) hyperbolic ray emanating from a vertex P bisecting the angle at P. Let the angle bisectors of α and β intersect at a point P inside T. Draw a hyperbolic line segment connecting P to C, and let this line segment divide the angle γ to angles γ_1 and γ_2. Let $v = d_{\mathbb{H}}(A,P)$, $w = d_{\mathbb{H}}(B,P)$, and $m = d_{\mathbb{H}}(C,P)$. Applying the hyperbolic lat of sines in triangles ACP, BCP, and ABP, we have

$$\frac{\sinh(m)}{\sin(\alpha/2)} = \frac{\sinh(v)}{\sinh(\gamma_2)}; \ \frac{\sinh(m)}{\sin(\beta/2)} = \frac{\sinh(w)}{\sinh(\gamma_1)}; \ \frac{\sinh(w)}{\sin(\alpha/2)} = \frac{\sinh(v)}{\sinh(\beta/2)}$$

These equations imply that

$$\frac{\sinh(m)}{\sin(\alpha/2)} = \frac{\sinh(\gamma_1)}{\sinh(\gamma_2)}\frac{\sinh(m)}{\sinh(\alpha/2)},$$

and so we have that $\sin(\gamma_1) = \sin(\gamma_2)$. As $0 < \gamma_1, \gamma_2 < \pi$, it follows that $\gamma_1 = \gamma_2$.

(b) Let the triangle T have internal angles α, β, and γ. By applying the hyperbolic law of sines, we have

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(a)}{\sin(\beta)} = \frac{\sinh(a)}{\sin(\gamma)},$$

from which the first part of (b) follows.

For the second part, consider a triangle T' with internal angles $\alpha/2$, α, and $\pi/2$ obtained from T by dropping an angle bisector from a vertex to the opposite side. Note that T' has side lengths a, $a/2$ and b. By applying the first hyperbolic cosine law in T' and simplifying, we have

$$\cos(\alpha/2) = 2\sin(\alpha/2)\cos(\alpha/2)\cosh(a/2),$$

which yields the desired result.

3. For $g \geq 2$, let P_{4g} be a compact regular hyperbolic $4g$-gon P_{4g} of unit side length.

   (a) Consider a decoration D of P_{4g} given by

   $$\partial(P_{4g}) = \prod_{i=1}^{2g} a_i \prod_{i=1}^{2g} a_i^{-1}.$$

   Show that D determines a hyperbolic structure X_g on S_g.

   (b) Consider another decoration of P_{4g} given by

   $$\partial(P_{4g}) = \prod_{i=1}^{g} [x_i, y_i].$$

   Show that D' too determines a hyperbolic structure X'_g on S_g.

Solution. (a) & (b) It is quite apparent that both decorated polygons described above satisfy the conditions of a gluing recipe, and will determine closed hyperbolic surfaces (why?). Furthermore, these surfaces will be orientable, as they do not contain an imbedded Möbius band (why?). In both cases, the identification induced by the decoration will yields a CW-complex structure on a closed orientable surface comprising 1 vertex, $2g$ edges, and 1 face. Consequently, by the classification of surfaces the hyperbolic surfaces obtained from these decorated polygons will be homeomorphic to S_k. It remains to show that $k = g$. However, this follows directly from the fact that

$$\chi(S_k) = 2 - 2k = 1 - 2g + 1.$$